Characterizing the Gröbner Bases of Generic Ideals

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Algebraic geometry is the study of finding the zeros of multivariate polynomials. Computational algebraic geometry then allows for the calculation of Gröbner bases, which are specific bases with desired properties used to characterize the ideal generated by a set of polynomials. This research was concerned exclusively with generic polynomials, which are homogeneous polynomials spanning all monomials of a given degree with coefficients that are algebraically independent over a field. Moreno-Socías conjectured that the initial ideal generated by generic polynomials relative to the graded reverse lexicographic ordering is weakly reverse lexicographic, meaning that in any degree stratum of the initial ideal, if x^α is a minimal generator, then every monomial of the same degree preceding x^α in grevlex order is also in the initial ideal. This conjecture has been proven up to $n = 3$ as well as for certain special cases. The goal of this research was to explicitly characterize the Gröbner bases in 3-variables so that it could then be extended to n-variables and used to prove Moreno-Socías conjecture works in the general case.

The early stages of the research process involved an intensive study of *Ideals, Varieties, and Algorithms* by Cox, Little, and O'Shea to understand the underlying mathematics in question. The research then progressed by looking at patterns found by previous researchers in initial ideals in the 3-variable case. The pattern was first confirmed and then manipulated to formulate a new conjecture used to characterize the Gröbner bases in 3 variables. The main method of research was trial and error, whereby variations of the conjecture were tested against test cases to see if there were holes in the conjecture. Furthermore, methods of proof were used to jump from the conjecture to Moreno-Socías conjecture and beyond.

To date, it has been shown that the new conjecture implies the Moreno-Socías conjecture in the 3-variable and consequently n-variable cases, and it has been confirmed to exist in the 2-variable case as well. By extension, this would mean that the conjecture implies Fröbergs conjecture on Hilbert series, but this has yet to be explicitly proven. The conjecture and test case analysis confirmed that the bases depend on the starting degrees of the generic polynomials, and furthermore, the Gröbner bases of any set of random generic polynomials can be constructed once the initial ideal is known.

Overall, I found this research to be a good introduction to the field of algebraic geometry which allowed me to learn quite a bit of new material. Through this experience, I got a glimpse of what graduate research will look like, and I discovered aspects of research that I appreciate and enjoy as well as others that are subtler and require more patience. This research confirmed my desire to pursue graduate studies in pure mathematics, and it still appears that my end goal will be higher education academia.