Characterizing the Gröbner bases of generic ideals

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An ideal is a valuable tool in analyzing polynomials and their roots. A Gröbner basis generates an ideal and has several desirable characteristics for working with that ideal. The initial ideal $in(I)$ is generated by the leading terms of the polynomials in the ideal I and describes the ideal I, but is it itself difficult to characterize. Moreno-Socías conjectured that the initial ideal generated by generic polynomials under the graded reverse lexicographic ordering is weakly reverse lexicographic. That is, within a given total degree, every monomial preceding a minimal generator of the initial ideal is contained in the initial ideal. Moreno-Socías proved the conjecture in 2-variables in his thesis, and the 2-variable and 3-variable cases have been proven through different methods in other papers. We focused on proving the 3-variable case through a new approach in hopes of generalizing the proof of the conjecture to n-variables.

We studied sections of *Ideals, Varieties, and Algorithms* by Cox, Little, and O'Shea to understand the Moreno-Socías conjecture and the methods used to prove it. We also studied different proofs for the 2-variable case in papers by Aguirre, Jarrah, Lauben-bacher, et al. and Aguirre, Ortiz-Navarro, and Torrez to gain insight into the 3-variable case. Furthermore, we reviewed the work done on this project by Kimberly Oesman and Isaac Zylstra in the summer of 2016, especially their characterization of the initial ideals generated by generic polynomials in 3-variables. We used a program in *Singular* written by Isaac Zylstra to generate Gröbner bases. Finally, we wrote programs in C++ to analyze the S-polynomials of our generated bases to work towards proving that our proposed bases were Gröbner bases.

After studying the previously generated bases, we refined the *Singular* program and generated more cases to study. We looked for patterns in the generators of the initial ideals and the ideals, and we refined the characterization of the initial ideals generated by generic polynomials in 3-variables and more explicitly characterized the resulting bases. This refinement came through trial and error; we tested specific cases of generated bases against versions of our conjecture to find errors in our conjecture. Then, we used induction to prove that our conjecture implies the Moreno-Socías conjecture in 3-variables and in n-variables. We also observed that our conjecture is true in the 2-variable case. Furthermore, we studied the S-polynomials that resulted from our generated bases and observed some patterns that may be useful in proving that our proposed bases are Gröbner bases.

This research project was meaningful to me because I had the opportunity to spend the summer immersed in the study of mathematics. I was able to explore algebraic geometry, a field of mathematics that was new to me, and I learned a lot through studying the background materials. I also gained practical experience working with different methods of mathematics research. Through working on this project, I experienced a taste of graduate-level research in mathematics. Thus, this summer has reinforced my aspiration to do graduate work in mathematics and explore more areas of mathematics.